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Inertial convection in rotating fluid spheres

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The onset of convection in the form of inertial waves in a rotating fluid sphere is studied through a perturbation analysis in an extension of earlier work by Zhang (1994). Explicit expressions for the dependence of the Rayleigh number on the azimuthal wavenumber are derived and new results for the case of a nearly thermally insulating boundary are obtained.

1. Introduction

Convection in the form of slightly modified inertial waves is a well-known phenomenon in geophysical fluid dynamics. The analysis of the onset of convection in a horizontal fluid layer heated from below and rotating about a vertical axis was first done by Chandrasekhar more than 50 years ago. For an account of this early work we refer to his famous monograph (Chandrasekhar 1961). He found that convection sets in at high rotation rates in the form of modified inertial waves when the Prandtl number is less than about 0.6 depending on the boundary conditions. Another important case in which convection in the form of modified inertial waves occurs is the rotating fluid sphere heated from within and subject to a spherically symmetric gravity field. The transition from convection in the form of columns aligned with the axis of rotation to inertial convection in the form of equatorially attached modes has been demonstrated by Zhang & Busse (1987). In a later series of papers Zhang (1993, 1994, 1995) developed an analytical theory for the critical parameter values for the onset of convection based on a perturbation approach. The buoyancy term and viscous dissipation are introduced in the equation of motion as small perturbations of inviscid inertial waves and the balance of the two terms is then used for the determination of the critical value of the Rayleigh number. In this paper we extend this approach to case of a spherical boundary of low thermal conductivity on the one hand and to an alternative method of analysis on the other hand which will allow us to obtain explicit expressions for the dependence of the Rayleigh number on the azimuthal wavenumber.

2. Mathematical formulation of the problem

We consider a homogeneously heated, self-gravitating fluid sphere rotating with the constant angular velocity Ω about an axis fixed in space. A static state thus exists with the temperature distribution $T_S = T_0 - \beta r_0^2 r^2/2$ and the gravity field given by $\boldsymbol{g} = -\gamma r_0 \boldsymbol{r}$ where $\boldsymbol{r}$ is the position vector with respect to the centre of the sphere and r is its length measured in fractions of the radius r_0 of the sphere. In addition to the length r_0, the time r_0^2/ν and the temperature $\nu^2/\gamma\alpha r_0^4$ are used as scales for the dimensionless description of the problem where ν denotes the kinematic viscosity of the fluid and κ is its thermal diffusivity. The density is assumed to be constant except in the gravity term where its temperature dependence given by $\alpha \equiv (\mathrm{d}\varrho/\mathrm{d}T)/\varrho =$ const. is taken into account. The basic equations of motion and the heat equation for the deviation Θ from the static temperature distribution are thus given by

$$\partial_t \boldsymbol{u} + \tau \boldsymbol{k} \times \boldsymbol{u} + \nabla \pi = \Theta \boldsymbol{r} + \nabla^2 \boldsymbol{u}, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1b}$$

$$R \boldsymbol{r} \cdot \boldsymbol{u} + \nabla^2 \Theta - P \partial_t \Theta = 0, \tag{2.1c}$$

where the Rayleigh number R, the Coriolis parameter τ and the Prandtl number P are defined by

$$R = \frac{\alpha \gamma \beta r_0^6}{\nu \kappa}, \quad \tau = \frac{2\Omega r_0^2}{\nu}, \quad P = \frac{\nu}{\kappa}. \tag{2.2}$$

We have neglected the nonlinear terms $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla \Theta$ in equations (2.1) since we restrict the attention to the problem of the onset of convection in the form of small disturbances. In the limit of high τ the right-hand side of equation (2.1*a*) can be neglected and the equation for inertial waves is obtained. For the description of inertial wave solutions $\boldsymbol{u}_0$ we use the general representation in terms of poloidal and toroidal components for the solenoidal field $\boldsymbol{u}_0$,

$$\boldsymbol{u}_0 = \nabla \times (\nabla v \times \boldsymbol{r}) + \nabla w \times \boldsymbol{r}. \tag{2.3}$$

By multiplying the (curl)^2 and the curl of the inertial wave equation by $\boldsymbol{r}$ we obtain two equations for v and w,

$$[\partial_t \mathscr{L}_2 - \tau \partial_\varphi] \nabla^2 v - \tau \mathscr{Q} w = 0, \tag{2.4a}$$

$$[\partial_t \mathscr{L}_2 - \tau \partial_\varphi] w + \tau \mathscr{Q} v = 0, \tag{2.4b}$$

where ∂_t and ∂_φ denote the partial derivatives with respect to time t and with respect to the angle φ of a spherical system of coordinates r, θ, φ and where the operators $\mathscr{L}_2$ and $\mathscr{Q}$ are defined by

$$\mathscr{L}_2 \equiv -r^2 \nabla^2 + \partial_r (r^2 \partial_r), \tag{2.5a}$$

$$\mathscr{Q} \equiv r \cos\theta \nabla^2 - (\mathscr{L}_2 + r\partial_r)(\cos\theta \partial_r - r^{-1} \sin\theta \partial_\theta). \tag{2.5b}$$

General solutions in explicit form for inertial waves in rotating spheres have recently been obtained by Zhang *et al.* (2001). Here only solutions of equations (2.4) for which v is symmetric with respect to the equatorial plane and does not possess a zero in its θ-dependence are of interest since only those are connected with the preferred modes for the onset of convection (Zhang 1994). These modes are given by

$$v_0 = P_m^m(\cos\theta) \exp\{\mathrm{i}m\varphi + \mathrm{i}\omega\tau t\} f(r), \quad w_0 = P_{m+1}^m(\cos\theta) \exp\{\mathrm{i}m\varphi + \mathrm{i}\omega\tau t\} g(r), \tag{2.6}$$

with

$$f(r) = r^m - r^{m+2}, \quad g(r) = r^{m+1} \frac{2\mathrm{i}m(m+2)}{(2m+1)(\omega_0(m^2+3m+2) - m)}, \tag{2.7a}$$

$$\omega_0 = \frac{1}{m+2}(1 \pm (1 + m(m+2)(2m+3)^{-1})^{1/2}). \tag{2.7b}$$

Before considering the full problem (2.1) we have to specify the boundary conditions. We shall assume a stress-free boundary with either a fixed temperature (case A) or a thermally insulating boundary (case B),

$$\boldsymbol{r}\cdot\boldsymbol{u} = \boldsymbol{r}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{u})/r^2 = 0 \quad \text{and} \quad \left.\begin{array}{ll}\Theta = 0 & \text{(case A)}\\ \partial_r\Theta = 0 & \text{(case B)}\end{array}\right\} \quad \text{at} \quad r = 1. \tag{2.8}$$

Following Zhang (1994) we use a perturbation approach for solving equations (2.1),

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1 + \ldots, \qquad \omega = \omega_0 + \omega_1 + \ldots. \tag{2.9}$$

The perturbation $\boldsymbol{u}_1$ consists of two parts, $\boldsymbol{u}_1 = \boldsymbol{u}_i + \boldsymbol{u}_b$, where $\boldsymbol{u}_i$ denotes the perturbation of the interior flow, while $\boldsymbol{u}_b$ is the Ekman boundary flow which is required since $\boldsymbol{u}_0$ satisfies the first of conditions (2.8), but not the second.

After the ansatz (2.9) has been inserted into equations (2.1a) and (2.1b) we obtain the solvability condition for equation (2.1a) for $\boldsymbol{u}_1$ by multiplying it with $\boldsymbol{u}_0^*$ and averaging it over the fluid sphere,

$$\mathrm{i}\omega_1\langle|\boldsymbol{u}_0|^2\rangle = \langle\Theta\boldsymbol{r}\cdot\boldsymbol{u}_0^*\rangle + \langle\boldsymbol{u}_0^*\cdot\nabla^2(\boldsymbol{u}_0+\boldsymbol{u}_b)\rangle, \tag{2.10}$$

where the brackets $\langle\ldots\rangle$ indicate the average over the fluid sphere and $*$ indicates the complex conjugate. The evaluation of the second term on the right-hand side of (2.10) yields

$$\langle\boldsymbol{u}_0^*\cdot\nabla^2(\boldsymbol{u}_0+\boldsymbol{u}_b)\rangle = \langle(\nabla\times\boldsymbol{u}_0^*)\cdot(\nabla\times\boldsymbol{u}_b))\rangle + \frac{3}{4\pi}\oint[\boldsymbol{u}_0^*\cdot\nabla\boldsymbol{u}_b - \boldsymbol{u}_0^*\cdot(\boldsymbol{r}\cdot\nabla)\boldsymbol{u}_b]\,\mathrm{d}^2S, \tag{2.11}$$

since $\nabla^2\boldsymbol{u}_0$ vanishes (Zhang 1994). Since $\boldsymbol{u}_b$ is of the order $\tau^{-1/2}$ and vanishes outside a boundary layer of thickness $\tau^{-1/2}$, only the term involving a radial derivative of $\boldsymbol{u}_b$ makes a contribution of order one on the right-hand side of equation (2.11). This term can easily be evaluated because of the condition $\boldsymbol{r}\cdot\nabla\boldsymbol{r}\times(\boldsymbol{u}_0+\boldsymbol{u}_b)/r^2 = 0$ at the surface of the sphere. Using expressions (2.6) and (2.7a) we thus obtain

$$\langle\boldsymbol{u}_0^*\cdot\nabla^2\boldsymbol{u}_b\rangle = \frac{3}{2}\int_{-1}^{1}\left|P_m^m\right|^2\mathrm{d}(\cos\theta)\,m(m+1)(2m+1)\left[4+(m+2)\frac{(2m+1)}{2m+3}\right] \times\left|\frac{2(m+1)^2-2}{(2m+1)(\omega_0(m+1)(m+2)-m)}\right|^2, \tag{2.12}$$

where the relationship

$$\int_{-1}^{1}\left|P_m^{m+1}\right|^2\mathrm{d}\cos\theta = \frac{(2m+1)^2}{2m+3}\int_{-1}^{1}\left|P_m^m\right|^2\mathrm{d}\cos\theta \tag{2.13}$$

has been used.

3. Explicit expressions in the limit $P\tau \ll 1$

The equation (2.1c) for Θ can most easily be solved in the limit of vanishing $\tau P\omega_0$. In this limit we obtain for Θ,

$$\Theta = P_m^m(\cos\theta)\exp\{\mathrm{i}m\varphi + \mathrm{i}\omega\tau t\}h(r), \tag{3.1}$$

with

$$h(r) = m(m+1)R\left(\frac{r^{m+4}}{(m+5)(m+4)-(m+1)m} - \frac{r^{m+2}}{(m+3)(m+2)-(m+1)m} - cr^m\right), \tag{3.2}$$

where the coefficient c is given by

$$c = \begin{cases} \dfrac{1}{(m+5)(m+4)-(m+1)m} - \dfrac{1}{(m+3)(m+2)-(m+1)m} & \text{(case A)}, \\ \dfrac{(m+4)/m}{(m+5)(m+4)-(m+1)m} - \dfrac{(m+2)/m}{(m+3)(m+2)-(m+1)m} & \text{(case B)}. \end{cases} \quad (3.3)$$

Since Θ is real ω_1 must vanish according to the solvability condition (2.10) and we obtain for R the final result

$$R_{\pm} = \left(\frac{m^2(m+2)^3}{(2m+3)[(m+1)(1 \pm \sqrt{(m^2+4m+3)/(2m+3)}) - m]^2} + 2m + 1 \right) \times (2m+9)(2m+7)(2m+5)^2(2m+3)^2/b, \quad (3.4)$$

where the two possibilities for the sign originate from the two possibilities for the sign in the expression (2.7b) for ω_0. The coefficient b takes the values

$$b = \begin{cases} (m+1)m(10m+27) & \text{(case A)}, \\ (m+1)(14m^2+59m+63) & \text{(case B)}. \end{cases} \quad (3.5)$$

Obviously, the lowest value of R is reached for $m = 1$ and the value R_+ for convection waves travelling in the retrograde direction is always lower than the value R_- for the prograde waves. Expression (3.4) is also of interest, however, in the case of spherical fluid shells when the ($m = 1$)-mode is affected most strongly by the presence of the inner boundary. Convection modes corresponding to higher values of m may then become preferred at onset since their r-dependence decays more rapidly with distance from the outer boundary according to relationships (2.7).

4. Solution of the heat equation in the general case

For the solution of equation (2.1c) in the general case it is convenient to use the Green's function method. The Green's function $G(r, a)$ is obtained as solution of the equation

$$[\partial_r r^2 \partial_r + (-\mathrm{i}\omega_0 \tau P r^2 - m(m+1))]G(r, a) = \delta(r-a), \quad (4.1)$$

which can be solved in terms of the spherical Bessel functions $j_m(\mu r)$ and $y_m(\mu r)$,

$$G(r, a) = \begin{cases} G_1(r, a) = A_1 j_m(\mu r) & \text{for } 0 \leqslant r < a, \\ G_2(r, a) = A j_m(\mu r) + B y_m(\mu r) & \text{for } a < r \leqslant 1, \end{cases} \quad (4.2)$$

where

$$\mu \equiv \sqrt{-\mathrm{i}\omega_0 \tau P}, \quad A_1 = \mu \left(y_m(\mu a) - j_m(\mu a) \frac{j_m(\mu)}{y_m(\mu)} \right), \quad (4.3a, b)$$

$$A = -\mu j_m(\mu a) \frac{y_m(\mu)}{j_m(\mu)}, \quad B = \mu j_m(\mu a). \quad (4.3c, d)$$

A solution of equation (2.1c) can be obtained in the form

$$h(r) = -\int_0^1 G(r, a) m(m+1)(a^m - a^{m+2}) a^2 \, \mathrm{d}a$$
$$= -\int_0^r G_2(r, a) m(m+1)(a^m - a^{m+2}) a^2 \, \mathrm{d}a - \int_r^1 G_1(r, a) m(m+1)(a^m - a^{m+2}) a^2 \, \mathrm{d}a. \quad (4.4)$$

Evaluation of these integrals for $m = 1$ yields the expressions

$$h(r) = \begin{cases} \dfrac{2R}{(\omega_0 \tau P)^2}\left(r(\mu^2+10) - \mu^2 r^3 - \dfrac{10(\mu r\cos(\mu r) - \sin(\mu r))}{r^2(\mu\cos\mu - \sin\mu)}\right) & \text{(case A)},\\[2ex] \dfrac{2R}{(\omega_0 \tau P)^2}\left(r(\mu^2+10) - \mu^2 r^3 - \dfrac{(\mu^2-10)(\mu r\cos(\mu r) - \sin(\mu r))}{r^2(2\mu\cos\mu - (2-\mu^2)\sin\mu)}\right) & \text{(case B)}. \end{cases} \tag{4.5}$$

Slightly more complex expressions are obtained for $m > 1$. Expressions (4.5) can now be used to calculate R and ω_1 on the basis of equation (2.1). In the case $m = 1$ we obtain

$$\begin{aligned} R = 21(\omega_0\tau P)^2 \left(1 + \frac{9}{5(6\omega_0 - 1)^2}\right) \\ \times \begin{cases} \left[2 - 1050\mu^{-4} - \mathrm{Re}\left\{\dfrac{350\mu^{-2}\sin\mu}{\mu\cos\mu - \sin\mu}\right\}\right]^{-1} & \text{(case A)},\\[2ex] \left[9 + 525\mu^{-4} - \mathrm{Re}\left\{\dfrac{(7\mu^2 - 70 + 175\mu^{-2})\sin\mu}{2\mu\cos\mu + (\mu^2-2)\sin\mu}\right\}\right]^{-1} & \text{(case B)}, \end{cases} \end{aligned} \tag{4.6}$$

where Re{} indicates the real part of the term enclosed by {}. Expressions (4.6) have been plotted together with the expressions obtained for higher values of m in figures 1(a) and 1(b) for the cases A and B, respectively. We also show by broken lines numerical values which have been obtained through the use of a modified version of the Galerkin method of Ardes, Busse & Wicht (1997). Because the numerical computations have been done for the finite value 10^5 of τ the results differ slightly from those of the analytical theory. Since there are two values of ω_0 for each m, two functions $R(\tau P)$ have been plotted for each m. For values τP of order unity or lower, expressions (3.4) are approached well and the retrograde mode corresponding to the positive sign in (2.7b) yields always the lower value of R. But the prograde mode corresponding to the negative sign in (2.7b) becomes the preferred mode as τP becomes of order 10 or larger depending on the particular value of m. This transition can be understood on the basis of the increasing difference in phase between Θ and u_r with increasing τP. While the mode with the largest absolute value of ω_0 is preferred as long as Θ and u_r are in phase, the mode with the minimum absolute value of ω becomes preferred as the phase difference increases since the latter is detrimental to the work done by the buoyancy force. The frequency perturbation ω_1 usually makes only a small contribution to ω, which tends to decrease the absolute value of ω.

For very large values of τP the Rayleigh number R increases in proportion to $(\tau P)^2$ for fixed m. In spite of this strong increase Θ remains of order τP on the right-hand side of equation (2.1a). The perturbation approach thus continues to be valid for $\tau \longrightarrow \infty$ as long as $P \ll 1$ can be assumed. For any fixed low Prandtl number, however, with increasing τ the onset of convection in the form of prograde inertial modes will be replaced at some point by onset in the form of columnar convection because the latter obeys an approximate asymptotic relationship for R of the form $(\tau P)^{4/3}$ (see, for example, Busse 1970). This second transition depends on the value of P and will occur at higher values of τ and R for lower values of P. There is little chance that inertial convection occurs in the Earth's core, for instance, since P is of the order 0.03 while the usual estimate for τ is 10^{15}.

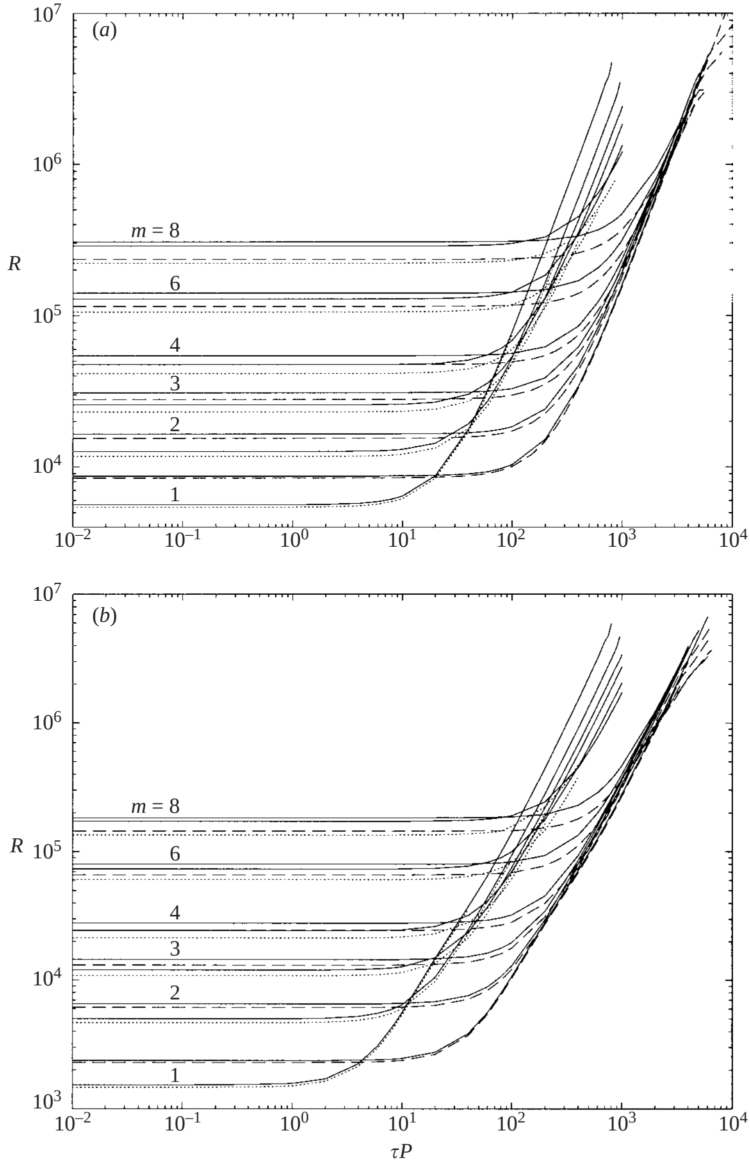

FIGURE 1. The Rayleigh number R as a function of τP for $m = 1, 2, 3, 4, 6$ and 8. Results based on explicit expressions such as (4.6) in the case of $m = 1$ (solid lines) are shown in comparison with the results obtained with a Galerkin numerical scheme (dotted lines for retrograde mode, dashed lines for prograde mode). (a) Case A, fixed temperature boundary conditions. (b) Case B, insulating thermal boundary conditions.

5. Discussion

Since the curves $R(\tau P, m)$ intersect at values of τP of order 10^3 in figures 1(a) and 1(b), a different way of plotting the results has been adopted in figure 2. Here the preferred value of m has been indicated by a filled circle in the case of the prograde inertial mode. The results of figure 2(a) agree well with those of figure 4 of Zhang (1994) even though only an approximate method had been used for the

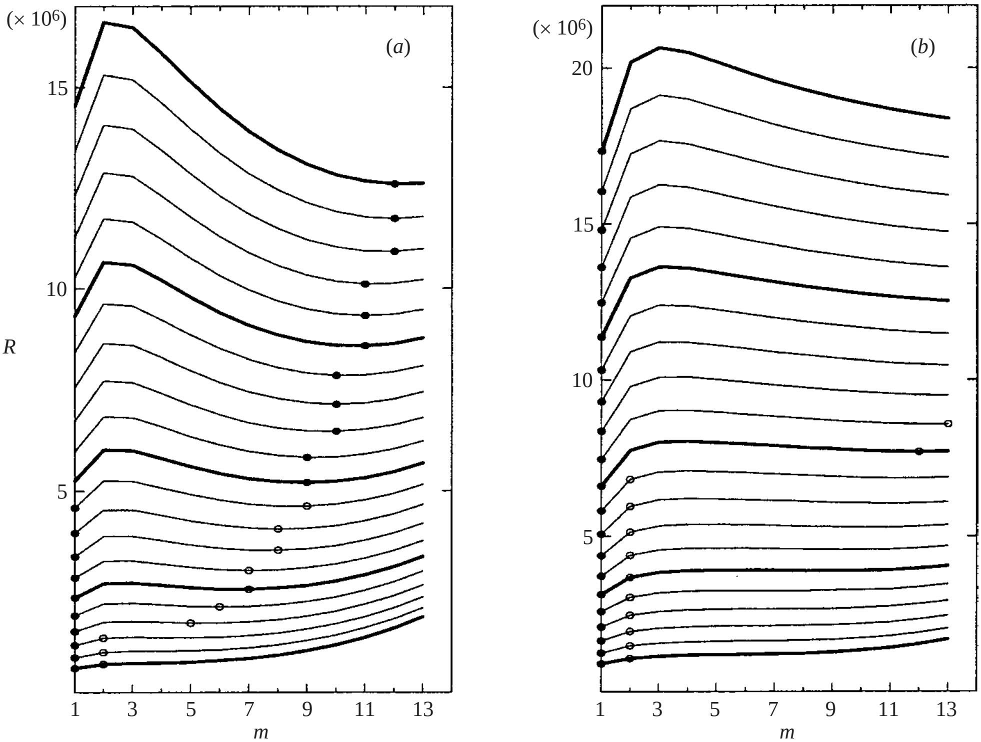

FIGURE 2. The Rayleigh number R as a function of m for $2 \times 10^3 \leqslant \tau P \leqslant 10^4$ (from bottom to top). The lines are equidistant with a step of $\Delta(\tau P) = 400$. The filled circles indicate the preferred values of m. The open circles correspond to the preferred value of m in the case when $m = 1$ is not included in the competition. (a) Case A, fixed temperature boundary conditions. (b) Case B, insulating thermal boundary conditions.

determination of the Rayleigh number. Zhang neglected the ($m = 1$)-mode and thus arrived at a different criterion for the preferred mode. His preferred values of m are indicated by open circles in figure 2. The ($m = 1$)-mode could indeed be suppressed by the presence of an inner concentric spherical boundary. A rough estimate indicates that inertial convection with azimuthal wavenumber m will be affected significantly when the radius η of the inner boundary exceeds a value of the order $(1 - m^{-1})$. Unfortunately, an analytical theory of inertial waves in rotating spherical fluid shells does not exist and it is thus not possible to extend the analysis of this paper to the case when an inner boundary is present. For a numerical study of inertial convection in rotating spherical fluid shells and its finite-amplitude properties we refer to Simitev & Busse (2003).

The two transitions between modes of different types mentioned in the preceding section illuminate some of the puzzling findings of Zhang & Busse (1987) and Ardes *et al.* (1997). The transition labeled I in figure 17 of Zhang & Busse (1987) can now be clearly identified with the transition from retrograde to prograde inertial convection. The main result of our analysis is that this transition depends primarily on the parameter combination τP with only a minor dependence on the wavenumber m. The second transition (1997). The second transition from inertial to columnar convection cannot be pinned down equally well because of the lack of a sufficiently accurate analytical theory for thermal Rossby waves in the low Prandtl number regime. According to the numerical results of Ardes *et al.* (1997) (see their figures 4 and 5) there exists a broad transition range involving perhaps several transitions where the onset of convection occurs in the form of multi-cellular modes. An illumination of this regime should be the goal of future research.

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